LYAPUNOV STABILITY OF
SOME TRANSFORMED EQUATIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "LYAPUNOV STABILITY OF SOME TRANSFORMED EQUATIONS", submitted by LEROY T. BEACH in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis deals with stability criteria and Lyapunov functions for certain second order differential equations.

The function $x'' + x = f(t,x,\varepsilon)$ is investigated for stability of its periodic motions by transforming the problem to an equivalent stability problem of equilibrium points.

In Chapter I, some properties such as singular points and criteria for their classification are reviewed. Definitions are given, Lyapunov functions are defined and Lyapunov theorems are stated.

Chapter II is devoted to the transformed Duffing equation. The stability of this equation is examined.

In Chapter III, the transformed Van der Pol equation is investigated. Here reference is made to the existence of a limit cycle. Both the phase plane and the Lienard plane are discussed.

In Chapter IV, the transformed simple pendulum equation is discussed first without damping and then with damping.

Finally some of the effects of perturbation on the stability of a system are discussed in an Appendix at the end of the thesis.

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TABLE OF CONTENTS

TABLES AND FIGURES

CHAPTER I

BASIC CONCEPTS

In this chapter we review topics pertaining to the development of the other chapters. Reference is made to specific texts where further details can be found.

1. Study of Singular Points.

Consider a second order system of the form

$$\frac{dx}{dt} = P(x,y)$$
$$\frac{dy}{dt} = Q(x,y). \tag{1.1}$$

Such a system in which the independent variable does not appear explicitly is called autonomous.

We shall assume that $P(x,y)$ and $Q(x,y)$ are defined in some domain D of the $x - y$ plane and satisfy a Lipschitz condition in both x and y in some neighbourhood of each point of D.

Hurewicz [6] has shown that if t_o is any real number and

(x_o, y_o) any point of D, there exists a unique solution of (1.1)

$$x = x(t) \quad y = y(t) \tag{1.2}$$

satisfying $x(t_o) = x_o \quad y(t_o) = y_o$.

Furthermore it is known that either

(i) The solutions (1.2) are defined for all real values of t or

(ii) If the solution (1.2) are not defined for $t > t_1$ (say), then either

(a) As $t \to t_1 - 0$, the point $(x(t), y(t))$ approaches the boundary of D or if D is unbounded, possibly

(b) As $t \to t_1 - 0$ either $x(t)$ or $y(t)$ or both become unbounded.

In any subdomain D' of D in which $P(x,y)$ does not vanish we may write (1.1) in the form

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \tag{1.3}$$

This is a restricted direction field and we know that through any point of D', there exists a unique integral curve of (1.3), for if two distinct integral curves were to have a point in common, they would have to be tangent at this point, a possibility ruled out by the fact that P and Q are Lipschitzian. Let us consider a solution (1.2) of (1.1) with the restriction that $x(t)$ and $y(t)$ are not both constants. Then (1.2) defines a curve with continuous

turning tangent. This curve is called a characteristic curve or a trajectory. Note there is a difference between solutions and trajectories of (1.1) : a trajectory is a curve in D that is represented parametrically by more than one solution. Thus $x(t)$, $y(t)$, $x(t+c)$, $y(t+c)$ represent distinct solutions but they represent the same curve parametrically.

We may think of (1.1) as a "flow" in the phase plane defined by a velocity vector field. Consider the vector field defined by $V(x,y) = (P(x,y),Q(x,y))$ that is at (x,y), the vector $V(x,y)$ has a horizontal component $P(x,y)$ and a vertical component $Q(x,y)$. Equation (1.1) defines the motion of a particle (x,y) in the plane by the condition that its velocity at each point be equal to the vector $V(x,y)$.

The above material is fully discussed in Hurewicz [6].

<u>Definition 1.4.</u> A singular or a critical point (x_o,y_o) is a stationary point of the flow $P(x_o,y_o) = Q(x_o,y_o) = 0$ and the integral curves passing through it consists of the point itself.

<u>Definition 1.5</u> A critical point (x_o,y_o) of (1.1) is called an isolated critical point if there exists a neighbourhood of (x_o,y_o) which contains no other critical points.

We now introduce the notion of stability of a critical point ~~on~~ or equivalently stability of the solution $x(t) = x_o$, $y(t) = y_o$ $-\infty < t < \infty$ of (1.1).

<u>Definition 1.6.</u> Let (x_o,y_o) be an isolated critical point of (1.1), then (x_o,y_o) is said to be stable if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

(i) every trajectory of (1.1) in the δ-neighbourhood of (x_o,y_o) for some $t = t_1$ is defined for $t_1 \leq t < \infty$ and

(ii) if a trajectory satisfies (i) it remains in the ε-neighbourhood of (x_o,y_o) for $t > t_1$.

If in addition every trajectory $C : x = x(t), y = y(t)$ satisfying (i) and (ii) also satisfies (iii) $\lim\limits_{t \to \infty} x(t) = x_o$ and $\lim\limits_{t \to \infty} y(t) = y_o$, then (x_o,y_o) is said to be asymtotically stable. Finally, an isolated critical point which is not stable is said to be unstable.

The definition of stability roughly states that (x_o,y_o) is stable if once a trajectory enters a small disc containing (x_o,y_o) it remains within a larger disc for all future time. The above definition is sometimes called stability to the right. A similar definition is given for stability to the left as $t \to -\infty$.

2. <u>Singularities of a linear system.</u>

In this section we consider the linear system

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy . \end{aligned} \tag{1.7}$$

where a,b,c,d are real constants. Therefore we may let D be the entire xy-plane and so all solutions are uniquely defined on $-\infty < t < \infty$. Hence we can discuss the behaviour of trajectories in the phase plane of (1.7). We discuss a system of the form (1.7) because a complete description of the phase plane can be given. Secondly many systems can be expressed in the form

$$x' = ax + by + \varepsilon_1(x,y)$$

$$y' = cx + dy + \varepsilon_2(x,y)$$

(the prime (') denotes differentiation with respect to t). If ε_1 and ε_2 are sufficiently small in a neighbourhood of a critical point, we hope that the behaviour of the trajectories is locally like that of (1.7). The system (1.7) has a singularity at (0,0). Assume it has no other singularity that is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

The characteristic equation of (1.7) is

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

where $\lambda = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$. Several cases arise depending on the value of the discriminant, $\Delta = (a-d)^2 + 4bc$. These cases will be discussed in a manner similar to that of Ritzer and Rose [9].

Case I. $\Delta = 0$. This could occur if $a = d \neq 0$ and $b = c = 0$. Thus $x' = ax$, $y' = ay$

$$x = c_1 e^{at} ; \quad y = c_2 e^{at}$$

Therefore $c_2 x = c_1 y$.

When all the solutions go through the singularity as in this case we have a node. If $a < 0$ as $t \to \infty$, x and $y \to 0$, the node is stable. If $a > 0$, it is unstable.

Case II.

$$\Delta > 0 \quad b = c = 0 , \quad a \neq d$$

We get

$$x' = ax \qquad y' = dy$$

$$x = c_1 e^{at} \qquad y = c_2 e^{dt}$$

Eliminating t, we get

$$c_4 y^a = c_3 x^d.$$

If a and d have the same sign we have a node. If a and d are of different signs so that when $x \to 0$, $y \to \infty$, when $x \to \infty$, $y \to 0$, we have a saddle point. Whether we have a node or saddle point, there are two special cases which are obtained when $c_3 = 0$ or when $c_4 = 0$. The curves are then $y = 0$ and $x = 0$. In the node, the curve $x = 0$ is the only one which enters the origin vertically. In the saddle the curves $x = 0$ and $y = 0$ are the only ones which enter the singularity.

Case III. $\Delta < 0$. This occurs if $d = a$ and $c = -b$. The differential equation becomes

$$x' = ax + by$$

$$y' = -bx + ay.$$

These equations have a solution

$$x = e^{at}\sin bt, \quad y = e^{at}\cos bt.$$

Hence

$$\sqrt{x^2 + y^2} = e^{at} .$$

If $a < 0$, the distance $\sqrt{x^2 + y^2}$ from the origin to a

point moving on the solution curve will decrease steadily as the point moves round the origin. In such a case, the singularity is called a stable spiral curve.

If $a > 0$, the curves spiral away from the origin and we have an unstable spiral.

In the above example, if $a = 0$, the solution would be $x = c_1 \sin bt \quad y = c_2 \cos bt$. The integral curves will be circles on more generally ellipses. In this case the origin is a centre.

Table 1 : Classification of Singularities

$$\frac{dy}{dx} = \frac{cx+dy}{ax+by} \quad \text{or} \quad \begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

$$\text{Fundamental quantities} \begin{cases} \Delta = (a-d)^2 + 4bc \\ p = a + d \\ q = ad - bc \neq 0 \end{cases}$$

Case	Singularity	Stability
(1) $\Delta = 0$	Node	Stable if $p < 0$; Unstable if $p > 0$
(2) $\Delta > 0$	Node if $q > 0$; Saddle if $q < 0$	Stable if $p < 0$; Unstable if $p > 0$
(3) $\Delta < 0$	Centre if $p = 0$; Spiral if $p \neq 0$	Stable if $p < 0$; Unstable if $p > 0$.

3. Stability for Non-Autonomous Systems.

Consider

$$x' = f(t,x) \tag{1.8}$$

where $x = x(t) = (x_1(t),\ldots,x_n(t))$ is an unknown n-dimensional vector function and assume that $f(t,x) = (f_1(t,x),\ldots,f_n(t,x))$ is defined and continuous in

$$D = \{(t,x) \mid r_1 < t < \infty, \ ||x|| < a\} .$$

By $||x||$ we mean $||x|| = (\sum_{i=1}^{n} |x|^2)^{\frac{1}{2}}$. A solution (not necessarily unique) of (1.8) satisfying $x(t_o) = x_o$ will be denoted by $x(t) = x(t,t_o,x_o)$.

<u>Definition 1.9</u> Let $x(t) = x(t,t_o,x_o)$ be a solution of (1.8) satisfying

(i) $x(t)$ is defined for $t_o \leq t < \infty$ and

(ii) the point $(t,x(t))$ belongs to D for $t \geq t_o$.

Then $x(t)$ is said to be stable if:

(a) There exists $\gamma > 0$ such that every solution $x(t;t_o,x_1)$ satisfies (i) and (ii), whenever $||x_1-x_o|| < \gamma$ and

(b) given $\varepsilon > 0$, there exists $\delta > 0$, $0 < \delta \leq \gamma$ such that $||x_o-x_1||<\delta$ implies.

$$||x(t,t_o,x_o)-x(t,t_o,x_1)|| < \varepsilon \quad \text{for} \quad t_o < t < \infty.$$

A solution that is not stable is said to be unstable.

Definition 1.10. The solution $x(t) = x(t,t_o,x_o)$ of (1.8) is asymtotically stable if it is stable and in addition there exists $\rho > 0$, $0 < \rho \leq \gamma$ such that $||x_o - x_1|| < \rho$ implies $\lim_{t \to \infty} ||x(t;t_o,x_o) - x(t;t_o,x_1)|| = 0$.

Geometrically, the definitions say that $x(t)$ is stable if any other solution whose initial data is sufficiently close to that of $x(t)$ remains in a "tube" enclosing $x(t)$. If the diameter of the tube approaches zero as $t \to \infty$, then $x(t)$ is asymtotically stable.

4. Lyapunov's Direct or Second Method.

We now discuss an important method of studying the equation (1.8) . The method is known as Lyapunov's direct or second method and involves constructing a particular type of function $V(t,x)$ from which the stability or instability of the solution in question can be determined.

Assume $f(t,x) = (f_1(t,x),\ldots,f_n(t,x))$ satisfies the following conditions

(i) $f(t,x)$ is defined and continuous in $D = \{(t,x) \mid ||x|| < a, r_1 < t < \infty\}$

(ii) a condition assuming uniqueness of solutions $x(t;t_o,x_o)$ is satisfied at every point (t_o,x_o) in D and

(iii) $f(t,0) = 0$ for all t and hence $x(t;t_o,0) = 0$ is a solution of (1.8) for $t_o > r$.

<u>Definition 1.11</u>. The class K consists of all continuous real valued strictly increasing functions $\phi(r)$, $0 \leq r \leq a$, which vanish at $r = 0$.

Let $0 < b \leq a$ and $t_o > r_1$, suppose $V(t,x)$ is a real valued function, continuous together with its first partial derivatives in the set

$$B = \{(t,x) \mid t_o \leq t < \infty, ||x|| \leq b\}$$

Furthermore assume $V(t,0) = 0$ for $t \geq t_o$.

<u>Definition 1.12</u>. The function $V(t,x)$ is positive (or negative) definite if there exists a function ϕ in the class K such that $V(t,x) \geq \phi(||x||)$ or $V(t,x) \leq -\phi(||x||)$ for all (t,x) in B.

<u>Definition 1.13</u>. The function $V(t,x)$ is said to be descrescent or to admit an infinitesimal upper bound if there exist $h > 0$ and a function ψ in the class K such that, $|V(t,x)| \leq \psi(||x||)$ for $||x|| < h$ and $t \geq t_o$.

By V' we denote the function

$$V' = V'(t,x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) + \frac{\partial V}{\partial t}$$

We will now state some theorems on stability, the proofs of

these may be found in Sánchez [10].

Theorem 1.14. If a function $V(t,x)$ exists that is positive definite and whose derivative V' is non positive, then the solution $x(t) \equiv 0$ of (1.8) is stable.

Theorem 1.15 If $V(t,x)$ exists that is positive definite, descrescent and whose derivative is negative definite, then the solution $x(t) \equiv 0$ of (1.8) is asymtotically stable.

Definition 1.16 A function $V(t,x)$ satisfying the hypotheses of Theorem 1.14, is called a Lyapunov function of equation (1.8).

In the case of the autonomous system, we omit the dependence of V on t and delete the family K which serves to ensure uniformity with respect to t. Thus $V = V(x)$ is a positive (negative) definite function if

(i) V is continuous together with its partial derivatives in some neighbourhood of the origin and

(ii) $V(x) \geq 0$ or (≤ 0) with equality when $x = 0$.

The two theorems in the autonomous case are much easier to prove. Now we will be dealing with the case $n = 2$. The curve $V(x,y) = c$ constitutes a family of concentric loops encircling the origin. The ovals grow in size as c increases from zero and each oval encloses others of smaller c.

The hypotheses of Theorem 1.14 now imply that given a critical

point at the origin and a suitable Lyapunov function in an open region R about it, all solutions that start within the oval $\mathcal{C}$ will remain in it.

Theorem 1.15 now implies that if $\mathcal{C}$, $V(x,y) = c$, does not cross the boundary of R, all solution trajectories that start at a point $P_o(x_o,y_o)$ will tend to the origin as $t \to \infty$.

We have spoken only of stability. A natural question to ask is, "When do we have instability?"

In the autonomous case we just need V to be positive definite. V' to be positive definite and in every neighbourhood N of the origin $V(x_o) > 0$ for $x_o \in N$. This gives instability.

5. We shall discuss the method we shall use in examining the stability of the Duffing, Van der Pol and simple pendulum equations (the equations with which we will be dealing).

Consider again equation (1.8) . If the unperturbed motion is defined as the solution $p(t,t_o,x_o)$ of (1.8), then its stability can always be reduced to the stability of the equilibrium. For this purpose, we set $x = y + p(t,t_o,x_o)$ in (1.8) and obtain, the differential equation

$$y' = f(y+p(t,t_o,x_o),t) - f(p(t,t_o,x_o),t) \qquad (1.17)$$

The right side of this equation is a function of y and t which

vanishes for $y = 0$. The stability behaviour of the solution $p(t,t_o,x_o)$ is equivalent to the stability behaviour of the equilibrium of (1.17). Equation (1.17) is called the differential equation of the perturbed motion.

The goal of this transformation is a simplification of the stability problem; however the new differential equation is often more complicated than the original one. So the differential equation of the perturbed motion for the solution of an autonomous differential equation is almost always non-autonomous.

In what follows, the new equation will be considered on its own merit. Certain conditions will be imposed on $p(t)$ to get suitable Lyapunov functions.

CHAPTER II

THE TRANSFORMED DUFFING EQUATION

1. In engineering equations of the type

$$mx'' + \phi(x') + f(x) = f(t) \tag{2.1}$$

are studied. The term mx'' is frequently referred to as the inertia force, $-\phi(x')$ as the damping force, $-f(x)$ as the restoring force or spring force and $f(t)$ as the external force. The terms in equation (2.1) are interpreted in many amny ways in the case of electrical and combined electrical-mechanical systems of various types.

The equation

$$x'' + x + \beta x^3 = f(t) \tag{2.2}$$

is an equation of this type and is called the Duffing equation (without damping). It is one of our purposes in this chapter to study the stability of motions of this equation. This is done by means of the transformation discussed in the last chapter.

Let $p(t)$ be a solution of (2.2). We use the transformation $y = x - p(t)$, giving $x = y + p(t)$.

Hence $x' = y' + p'(t)$, $x'' = y'' + p''(t)$. Substituting into equation (2.2), we have

$$y'' + p''(t) + y + p(t) + \beta(y^3+3y^2p(t)+3yp^2(t)+p^3(t)) = f(t).$$

Therefore

$$y'' + y + \beta(y^3+3y^2p(t)+3yp^2(t))$$

$$= - p''(t) - \beta p^3(t) - p(t) + f(t) = 0. \qquad (2.3)$$

Now $y \equiv 0$ is a solution of equation (2.3) corresponding to $x = p(t)$ and hence the problem of stability is reduced to studying stability of equilibrium points.

It is therefore natural to investigate the stability of the origin of the transformed equation, including those conditions that should be put on $p(t)$ to obtain results concerning the nature of the stability.

Actually, we will be trying to find suitable Lyapunov functions.

Without loss of generality, let us consider the equation

$$x'' + x + \beta(x^3+3x^2p(t)+3xp^2(t)) = 0 \qquad (2.4)$$

instead of equation (2.3).

Consider first the simpliest form of this equation, the case when $p(t) \equiv 0$. Then equation (2.4) becomes

$$x'' + x + \beta x^3 = 0 . \tag{2.5}$$

In the phase plane, this becomes

$$\begin{aligned} x' &= y \\ y' &= -x - \beta x^3 \end{aligned} \tag{2.6}$$

Now, a Lyapunov function can be thought of as an "energy" function. A suitable Lyapunov function for equation (2.6) is

$$V(x,y) = \frac{1}{2} y^2 + \frac{1}{2} x^2 + \frac{1}{4} \beta x^4 = h . \tag{2.7}$$

There are two cases (non linear) to consider, $\beta > 0$ and $\beta < 0$, referred to as the hard and soft spring case respectivley. For the case $\beta = 0$, $V(x,y) = \frac{1}{2} x^2 + \frac{1}{2} y^2$ is a Lyapunov function. In this case we have stability in the whole plane -- this shall be referred to this thesis as global stability.

<u>Hard Spring $\beta > 0$.</u>

The origin is the only critical point of the system. V(x,y)

given by (2.7) is the Lyapunov function we use,

$$\frac{dV}{dt} = yy' + (x^2+\beta x^3)x' = 0 \ .$$

We again have in this case global stability. The trajectories in the phase plane are ovals symmetric about the origin. The linear approximation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ,$$

shows that the origin is a centre and the ovals are traversed counter clockwise. This is discussed in Brand [2].

Soft Spring $\beta < 0$.

Equation (2.6) has three critical points $(0,0)$ and $(0,\pm x_1)$ where $x_1 = \sqrt{-1/\beta}$.

The origin is still a centre but we do not have stability in the whole plane (global stability) for the points $P_o = (0,x_1)$ and $P_1 = (0,-x_1)$ are now saddle points. If we shift the origin to P_o by putting $x = X + x_1$ and $y = Y$ we get

$$X' = Y$$

$$Y' = -(X+x_1) - \beta(X+x_1)^3$$

whose linear approximation is

$$X' = Y$$
$$Y' = -(1+3\beta x_1^2)X = 2X.$$

From the table in Chapter 1, we see that this is a saddle point.

2. Curves of constant energy

We will discuss curves of constant energy in the (x,y) plane. The closed energy curves are integral curves corresponding to periodic motion $x(t)$. If $x(t)$ is periodic, the corresponding $x - y$ curve is closed. Conversely if any $x - y$ curve is closed, it follows that the displacement and velocity at any time t are reached again after a certain time T, that is, $x(t+T) = x(t)$. Hence the motion is periodic.

In a neighbourhood of the origin the curves given by equation (2.7) are closed curves which have the appearance of ovals since $\beta \frac{x^4}{4}$ can be neglected in comparison with $\frac{1}{2}x^2$ for small x. See Stoker [11].

In the case of the soft spring $\beta < 0$,

$$V(x,y) = y^2 + x^2 - k^2\frac{x^4}{2} = h, \quad k^2 = -\beta.$$

$$y^2 = h - x^2 + \frac{k^2x^4}{2} \tag{2.8}$$

The right-hand side of equation (2.8) is a quadratic in x^2 with discriminent $1 - 2k^2h$; it is therefore positive if $1 - 2k^2h$ is negative and in this case h must of necessity be positive. The transition from curves which cross the x-axis to curves which do not, occurs at $1 - 2k^2h = 0$ that is at $h = \frac{1}{2k^2}$.

On this curve the velocity y_o corresponding to $x = 0$ has the value $y_o = \frac{1}{k\sqrt{2}}$. So for $y_o > \frac{1}{k\sqrt{2}}$ we have open curves, $y_o < \frac{1}{k\sqrt{2}}$ we have closed curves encircling the origin. The transition curve corresponding to $y_o = \frac{1}{k\sqrt{2}}$ has two double points on the x-axis at $x = \pm\sqrt{-1/\beta} = \pm\sqrt{1/k^2} = \pm 1/k$. This curve separates the $x - y$ plane into regions in which three different types of curves occur. See Figure 2.1 below.

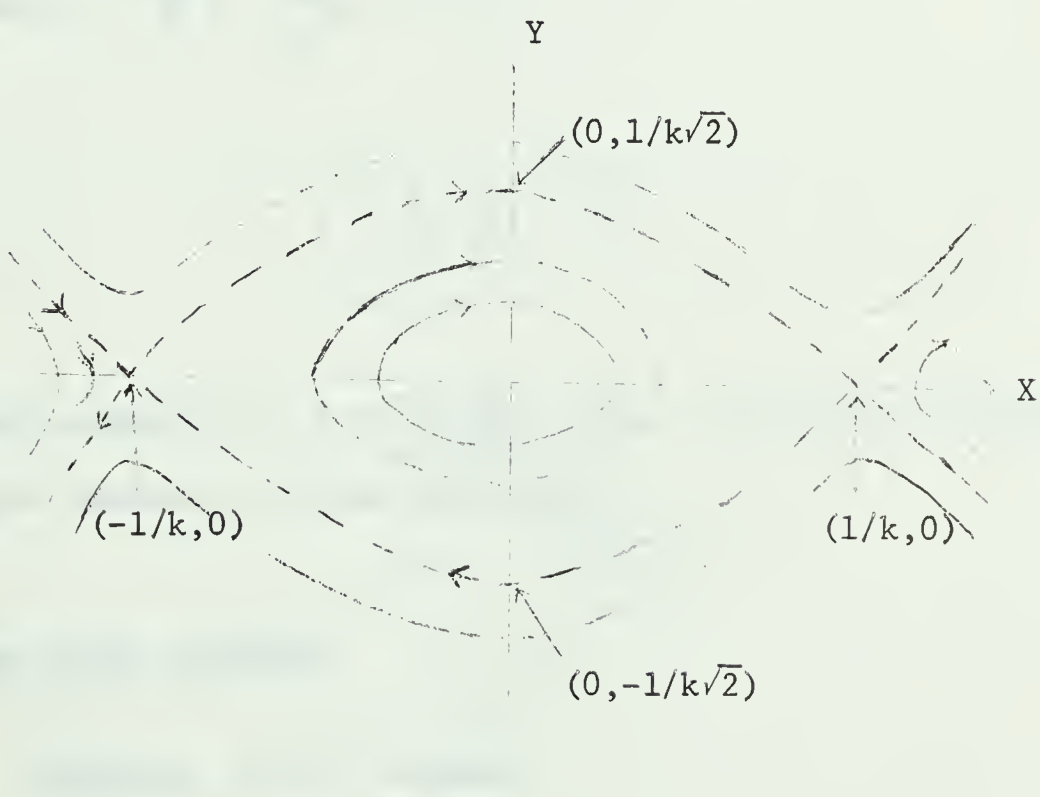


Figure 2.1.

Figure 2.1 shows two sets of curves which represent non-periodic motion and one set of curves which represent periodic motion.

We can show that the separatrices for the soft spring are parabolas. The separatrices are the "dotted" curves in the figure.

$$A(y - \frac{1}{k\sqrt{2}}) = x^2$$

$$A(0 - \frac{1}{k\sqrt{2}}) = \frac{1}{k^2}$$

$$A = - \frac{\sqrt{2}}{k} .$$

So we have $(- \frac{\sqrt{2}}{k} y - \frac{1}{k^2}) = x^2$ or

$$y = \frac{k}{\sqrt{2}} (\frac{1}{k^2} - x^2) .$$

The other parabola is $y = \frac{-k}{\sqrt{2}} (\frac{1}{k^2} - x^2)$. The region of stability is the region bounded by these two curves.

3. <u>Case p(t) constant</u>.

Equation (2.4) becomes

$$x'' + x + \beta(x^3+3px^2+3p^2x) = 0 \quad \text{or}$$

$$x' = y$$

$$y' = -x - \beta(x^3+3px^2+3p^2x). \tag{2.9}$$

Let $V(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \beta(\frac{x^4}{4} + px^3 + \frac{3}{2}p^2x^2) = h_1$. This is a Lyapunov function, $V'(x,y) \equiv 0$. Thus for the case $\beta > 0$, we again have stability, the origin is a centre. When $\beta < 0$, we claim that we have two saddle points as in the case $p(t) \equiv 0$. We will also examine the region of stability if one exists. One critical point is the origin, we can find the other two critical points by setting $y' = 0$.

Then

$$\beta x^3 + 3p\beta x^2 + 3p^2\beta x + x = 0$$

$$x(\beta x^2+3p\beta x+3p^2\beta+1) = 0$$

$$x(x^2+3px+\frac{3p^2\beta+1}{\beta}) = 0$$

$$x = \frac{-3p \pm \sqrt{9p^2 - \frac{4(3p^2\beta+1)}{\beta}}}{2}$$

$$= \frac{-3p \pm \sqrt{\frac{-3p^2\beta-4}{\beta}}}{2}.$$

Hence the other two critical points are

$$P_o = \left(-\frac{3}{2}p - \sqrt{\frac{-3}{4}p^2 - \frac{1}{\beta}}\,,\ 0\right) \text{ and}$$

$$P_1 = \left(-\frac{3}{2}p + \sqrt{\frac{-3}{4}p^2 - \frac{1}{\beta}}\,,\ 0\right).$$

Here we must stipulate that

$$-\frac{3}{4}p^2 - \frac{1}{\beta} > 0$$

i.e.

$$-\frac{3}{4}p^2 > \frac{1}{\beta}$$

or

$$\frac{3p^2}{4} < -\frac{1}{\beta}$$

$$-\frac{4}{3p^2} < \beta < 0 \quad .$$

Let us examine the system (2.9) by putting $\beta = -\frac{m}{p^2}$ $(\frac{4}{3} > m > 0)$.

We find that

$$x' = y$$

$$y' = -x + \frac{m}{p^2}(x^3+3px^2+3p^2x)$$

$$= -x + \frac{mx^3}{p^2} + \frac{3mx^2}{p} + 3mx$$

$$= (3m-1)x + \frac{mx^3}{p^2} + \frac{3mx^2}{p} \ .$$

Examining

$$\begin{pmatrix} 0 & 1 \\ (3m-1) & 0 \end{pmatrix}$$

the linear part of the system, we see that

$$\begin{cases} \Delta = 4(3m-1) \\ q = -(3m-1) \end{cases}$$

when $\Delta > 0$ i.e. $m > \frac{1}{3}$ $(\frac{-4}{3p^2} < \beta < -\frac{1}{3p^2})$, we have a saddle point at the origin. When $m < 1/3$ $(0 > \beta > -\frac{1}{3p^2})$ we have a centre at the origin. For $m > \frac{1}{3}$

$$P_o = (-\frac{3}{2}p - \sqrt{-\frac{3}{4}p^2 + \frac{p^2}{m}}, 0)$$

$$> (-\frac{3}{2}p - \sqrt{-\frac{3}{4}p^2 + 3p^2}, 0)$$

$$> (-3p, 0).$$

Thus $(-\frac{3}{2}p - \sqrt{-\frac{3}{4}p^2 + \frac{p^2}{m}}, 0) = (-3p + \delta_m, 0)$ (δ_m is some small positive number depending on m). Similarly

$$P_1 = (-\frac{3}{2}p + \sqrt{-\frac{3}{4}p^2 + \frac{p^2}{m}}, 0)$$

$$= (-\delta_m, 0).$$

When $m < \frac{1}{3}$

$$P_o = (-\frac{3}{2}p - \sqrt{-\frac{3}{4}p^2 + \frac{p^2}{m}}, 0)$$

$$= (-3p - \delta_m, 0)$$

$$P_1 = (-\frac{3}{2}p + \sqrt{-\frac{3}{4}p^2 + \frac{p^2}{m}}, 0) = (\delta_m, 0)$$

We can now examine the singularities at each of these points. The point $(-\delta_m, 0)$ terms out to be a centre and the other three points $(-3p+\delta_m, 0)$, $(-3p-\delta_m, 0)$ and $(\delta_m, 0)$ are all saddle points.

In conclusion we can say that the regions of stability is found wherever we have a centre. The two figures given below should be useful.

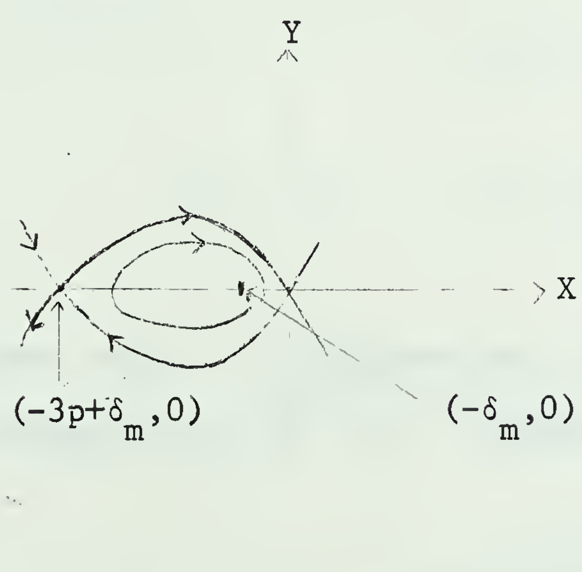


Figure 2.2.

$$-\frac{4}{3p^2} < \beta < -\frac{1}{3p^2}$$

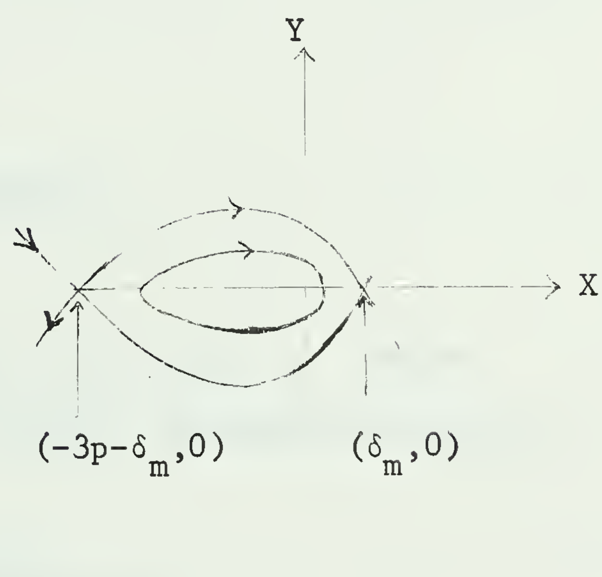


Figure 2.3.

$$0 > \beta > -\frac{1}{3p^2}$$

In the light of the preceeding discussion we get the following theorem.

Theorem 2.12. $V(x,y) = \frac{1}{2} y^2 + \frac{1}{2} x^2 + \beta(\frac{1}{4} x^4 + px^3 + \frac{3}{2} p^2x^2)$ is a suitable Lyapunov function for the system (2.9) with $\beta > 0$. For each value of p and for $\frac{-4}{3p^2} < \beta < -\frac{1}{3p^2}$, we have a saddle point at the origin of the corresponding system. The other two singularities of the system are a centre and a saddle. For $0 > \beta > -\frac{1}{3p^2}$, we have a centre at the origin, the other two singularities are saddles.

4. The case $p(t)$ non constant.

$$x' = y$$

$$y' = -(1+3\beta p^2(t))x - \beta(x^3+3p(t)x^2) .$$

Let $A(t) = 1 + 3\beta p^2(t)$, we will first examine the system

$$\begin{aligned} x' &= y \\ y' &= -(A(t))x \end{aligned} \tag{2.14}$$

with $A(t) > 0$ and then with $A(t) < 0$. $A(t) > 0$ implies $(1+3\beta p^2(t)) \geq \delta^2 > 0$. Equation (2.14) can now be written as

$$x'' + (1+3\beta p^2(t))x = 0 .$$

Multiplying by $x' = \frac{dx}{dt}$ we get

$$x'' \frac{dx}{dt} + (1+3\beta p^2(t))x \frac{dx}{dt} = 0 .$$

Integrating we get

$$(\frac{x'}{2})^2 + \int (1+3\beta p^2(t))xx'dt = C_1 .$$

But $(1+3\beta p^2(t)) \geq \delta^2$ and $x' = y$. Therefore

$$(\frac{y}{2})' + \int \delta^2 xx'dt \leq (\frac{y}{2})^2 + \int (1+3\beta p^2(t))xx'dt = C_1$$

$$\frac{y^2}{4} + \delta \frac{x^2}{2} \leq C_1 \qquad (2.15)$$

It is clear that for x and y satisfying (2.15) we have ellipses surrounding the origin. In this case the origin is a centre and we have stability.

When $A(t) < 0$, $(1+3\beta p^2(t)) \leq - \delta^2 < 0$. We then get

$$\frac{y^2}{4} + \int (1+3\beta p^2(t))xx'dt = C_2$$

$$\leq \frac{y^2}{4} - \delta x^2 \qquad (2.16)$$

Equation (2.16) represents hyperbolas and we have a saddle point at the origin -- an unstable situation. We state the following simple

theorem.

Theorem 2.17. Given system (2.14), we have stability at the origin if $A(t) > 0$ and instability if $A(t) < 0$.

The system (2.13) as a whole will now be studied, we will try to find a suitable Lyapunov function that will give us some information concerning the nature of the stability. Consider

$$V(x,y,t) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{3}{2}\beta p^2(t)x^2 + \frac{\beta}{4}x^4. \tag{2.18}$$

(We first look at the case $\beta < 0$).

Now (i) $V(x,y,t)$ is defined for all $t \geq 0$,

(ii) $V(0,0,t) = 0$ for $t \geq 0$.

(iii) $V(x,y,t) \geq W(x,y)$.

$W(x,y)$ will now be specified. Let $0 < p(t) < M$. Then

$$V(x,y,t) \geq \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{3}{2}\beta M^2x^2 + \frac{\beta}{4}x^4. \tag{2.19}$$

$W(x,y)$ is to be positive definite. Thus it should be that

$$(1+3\beta M^2) > 0$$

$$3\beta M^2 > -1$$

$$M^2 < -\frac{1}{3\beta}\ .$$

The right hand side of (2.19) is of the form

$$Ax^2 + \frac{1}{2}y^2 + \beta\frac{x^4}{4} \quad \text{where} \quad A = \frac{1}{2}(1+3\beta M)$$

and since this is to be our $W(x,y)$, it is necessary that

$$Ax^2 + \beta\frac{x^4}{4} > 0$$

$$x^2\left(A + \beta\frac{x^2}{4}\right) > 0$$

$$\frac{x^2}{4} < -\frac{A}{\beta}$$

$$x^2 < -\frac{4A}{\beta}$$

$$-\sqrt{\frac{-4A}{\beta}} < x < \sqrt{\frac{-4A}{\beta}}$$

So for the case $\beta < 0$, the region in which $V(x,y,t)$ is defined must be such that

$$-\sqrt{-\frac{2}{\beta}(1+3\beta M^2)} < x < \sqrt{-\frac{2}{\beta}(1+3\beta M^2)}\ .$$

For stability we want $V'(x,y,t) \leq 0$.

$$V'(x,y,t) = xx' + yy' + 3\beta p^2(t)xx' + 3\beta p'(t)p(t)x^2 + \beta x^3x'$$

$$= xy - xy - 3\beta p^2(t)xy - \beta x^3 y - 3\beta p(t)x^2 y$$

$$+ 3\beta p^2(t)xy + 3\beta p(t)p'(t)x^2 + \beta x^3 y$$

$$= 3\beta x^2(p'(t)p(t)-p(t)y) \quad .$$

We want $(p'(t)p(t) - p(t)y) \geq 0$

$$p'(t) \geq y \quad .$$

We will show that under the assumption $0 < p(t) < M$ the origin is unstable. For stability at the origin we should look at a neighbourhood of the origin say $|x| < \delta$, $|y| < \delta$. Thus we have

$$p'(t) \geq \delta \quad .$$

This implies that $p(t)$ becomes unbounded. So for negative β we are unable to find appropriate assumptions on $p(t)$ under which the origin is stable.

For $\beta > 0$, the positive definite function is $W(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Note in this case the positive definite function was easy to find in the sense that no restrictions had to be placed on x .

We will consider $p(t) \geq \delta > 0$. For stability

$$p'(t)p(t) - p(t)y \leq 0$$

i.e. $$p'(t) \leq y \quad .$$

Since we are concerned with stability at the origin, for $y \leq \delta$, $p'(t) \leq \delta$ and thus contradicts the fact that $p(t) > 0$. If $p(t) \leq -\delta < 0$, for stability

$$p'(t) \geq y \quad .$$

Thus $p'(t) \geq \delta$ and contradicts $p(t) < 0$. We can also see that for positive β, but are unable to find appropriate assumptions on $p(t)$ under which the origin is stable.

CHAPTER III

THE TRANSFORMED VAN DER POL EQUATION

1. Historical Note.

The Van der Pol equation

$$x'' + \mu(x^2-1)x' + x = 0$$

arose in the study of oscillatory vacuum tube circuits. Van der Pol showed that certain properties of vacuum tubes could be predicted only if the non linear terms of the equation were retained. The linear theory predicted that no periodic oscillations in the current would occur under certain conditions when experimental evidence showed they did occur. By retaining non-linear terms in the equation, Van der Pol was able to give a complete explanation of the phenomenon.

2. We will be considering the equation

$$x'' + \mu(x^2-1)x' + x = f(t). \tag{3.1}$$

Let $x = p(t)$ be a solution. Using the tranformation $y = x - p(t)$ or $x = y + p(t)$ we get $x' = y' + p'(t)$ and $x'' = y'' + p''(t)$. Substituting into equation (3.1) we have

$$y'' + p''(t) + \mu(y^2+2p(t)y+p^2(t)-1)(y'+p'(t)) + y + p(t) = f(t) \tag{3.2}$$

Now since $p(t)$ is a solution

$$p''(t) + \mu(p^2(t)-1)p'(t) + p(t) - f(t) = 0 \ .$$

Hence we get the resulting equation

$$y'' + \mu(y^2+2p(t)y+p^2(t)-1)y' + (\mu p'(t)y+2\mu p(t)p'(t)+1)y = 0 \ .$$

This is the equation we will study. Without loss of generality, let us consider

$$x'' + \mu(x^2+2p(t)x+p^2(t)-1)x' + (\mu p'(t)x+2\mu p(t)p'(t)+1)x = 0. \tag{3.3}$$

First we consider the case when $p(t) \equiv 0$, then $p(t) \equiv p$, a constant and lastly $p(t)$ as a non-constant function of t. Before doing this, some material pertaining to the understanding of this chapter will be introduced.

3. We consider the concept of a limiting set. This was first introduced by G.D. Birkoff.

Let

$$x' = X(x) \ , \quad X(0) = 0 \ . \tag{3.4}$$

Intuitively, if $x(t)$ is a solution of equation (3.4), its positive limiting set Γ^+ is whatever the curve $x(t)$ tends to with infinite time. Thus if $x(t)$ spirals around δ, a limit cycle, δ is its positive limiting set. (In the plane a limit cycle is an isolated closed path corresponding to a periodic solution). If it tends to a point A, the point is its positive limiting set.

More accurately p is in Γ^+ if there is an increasing sequence of times $t_n \to \infty$ with n such that $x(t_n) \to p$ as $n \to \infty$. If $x(t)$ is bounded, then $x(t)$ approaches its positive limiting set Γ^+ as $t \to \infty$, that is, given any $\varepsilon > 0$, if $N(\varepsilon)$ is the ε-neighbourhood of Γ^+, then there is a time T such that for $t > T$, $x(t)$ lies in $N(\varepsilon)$. This definition is due to LaSalle and Lefschetz [7]. We now give another definition.

<u>Definition 3.5</u>. An invariant set G is characterized by the property that if a point x_o is in G then its whole path (forward and backward) lies in G.

From the definition of limiting set there follow also the property: if $x(t)$ is bounded for $t \geq 0$ and if a set M contain Γ^+, then $x(t)$ tends to M as $t \to \infty$.

4. The Van der Pol equation is a special case of the equation

$$x'' + f(x)x' + g(x) = 0$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

which is known as the Lienard equation. This equation has been shown to have a unique stable limit cycle under the following conditions,

(a) $f(x)$ even, $g(x)$ odd, both continuous for all x and $f(0) < 0$.

(b) $xg(x) > 0$ for $x \neq 0$.

(c) For every interval $|x| < k$, there is an h such that

$$|g(x_1)-g(x_2)| < h|x_1-x_2|, \text{ a Lipschitz condition}$$

(d) $F(x) = \int_0^x f(x)dx \to \infty$ as $x \to \infty$.

(e) $F(x)$ has a single zero at $x = a$ and is monotone increasing for $x \geq 0$. For a proof of this theorem one can see Cesari [4].

LaSalle and Lefshetz [7] have proven two theorems which deal with the extent of asymtotic stability of the Lienard equation. We state these theorems.

Theorem 3.6 Let $V(x)$ be a scalar function with continuous first partial derivatives. Let Ω_ℓ designate the region where $V(x) < \ell$. Assume that Ω_ℓ is bounded and that within Ω_ℓ

(a) $V(x) > 0$ for $x \geq 0$

(b) $V'(x) \leq 0$.

Let R be the set of all point within Ω_ℓ where $V'(x) = 0$. Let M be the largest invariant set in R. Then every solution $x(t)$ in Ω_ℓ tend to M as $t \to \infty$.

Theorem 3.7. If condition (b) of the above theorem is replaced by $V'(x) < 0$ for all $x \neq 0$ in Ω_ℓ then the origin is asymtotically stable and above all every solution in Ω_ℓ tends to the origin as $t \to \infty$.

By considering the function

$$V(x,y) = \frac{1}{2} y^2 + G(x)$$

as a Lyapunov function for the Lienard equation and putting conditions on $G(x), x, g(x)F(x)$, they showed that the region Ω_ℓ defined by $V < \ell$ was a measure of asymtotic stability.

The Lienard equation

$$x'' + f(x)x' + g(x) = 0$$

is equivalent to

$$x'' = y - F(x) \;, \quad y' = -g(x)$$

for differentiating we get

$$x'' = y' - F'(x)x' = g(x) - f(x)x',$$

When $y = x' + F(x)$, the $x - y$ plane is called the Lienard plane.

Except when $f(x) = 0$, it differs from the phase plane in which $y = x'$.

5. We return to equation (3.3) and the case $p(t) \equiv 0$. We now get

$$x'' + \mu(x^2-1)x' + x = 0.$$

This is the Lienard equation in which

$$f(x) = \mu(x^2-1) \quad , \quad F(x) = \mu(\tfrac{1}{3}x^3-x)$$

$$g(x) = x .$$

The stability of the solutions depends on the sign of μ.

Case 1. $\mu > 0$. The phase plane equations

$$\begin{aligned} x' &= y \\ y' &= -x - \mu(x^2-1)y \end{aligned} \tag{3.8}$$

have a critical point $(0,0)$. When $x^2 < 1$

$$V(x,y) = x^2 + y^2$$

is a Lyapunov function, for

$$V'(x,y) = 2(xx'+yy') = -2\mu(x^2-1)y^2 < 0 .$$

The interior of the circle $x^2 + y^2 = 1$ is a region of asymtotic stability. Any trajectory that starts at a point within the circle tend to the origin. The linear part of (3.8) has the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} .$$

From the table in Chapter I,

$$\Delta = \mu^2 - 4$$
$$p = \mu$$
$$q = 1 .$$

So for

$\mu < -2$ the origin is a stable node.

$\mu = -2$ the origin is a stable node (inflected),

$-2 < \mu < 0$ the origin is a stable spiral.

<u>Case II. $\mu > 0$</u>

Now $f(0) = -\mu < 0$. The conditions for the existence of a stable limit cycle are satisfied. The origin is now unstable. The trajectories for any postive μ are same for $-\mu$ reversed in sense.

All trajectories inside the limit cycle tend to merge with it as $t \to \infty$. The limit cycle encloses the circle $x^2 + y^2 = 1$ in the phase plane.

We will now consider the situation in the Lienard plane,

$$x' = y - \mu(\frac{1}{3}x^3 - x), \quad y' = -x .$$

If $\mu < 0$, $V(x,y) = \frac{1}{2}(x^2+y^2)$ is a Lyapunov function for

$$V'(x,y) = \mu x^2(1-\frac{1}{3}x^3) < 0$$

when $x^2 < 3$. All paths within the circle $x^2 + y^2 = 3$ tend towards the origin. For corresponding positive $|\mu|$, all such paths are reversed in sense, cross the circle and eventually merge with the limit cycle.

For all positive values of μ, the limit cycle of the Van der Pol equation in the Lienard plane enclose the circle $x^2 + y^2 = 3$.

Brand [2] discusses the Lienard equation.

6. <u>The case $p(t) \equiv p$ (a constant).</u>

Equation (3.3) now has the form

$$x'' + \mu((x+p)^2-1)x' + x = 0$$

or

$$x' = y$$

$$y' = -\mu((x+p)^2-1)y - x \quad . \tag{3.9}$$

We want to study stability of this system.

Let $V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Then

$$V'(x,y) = xx' + yy'$$

$$= xy - \mu((x+p)^2-1)y^2 - xy$$

$$= -\mu((x+p)^2 - 1) \ .$$

For $\mu < 0$, $V' \leq 0$ if $(x+p)^2 \leq 1$. This implies $-1 \leq x + p \leq 1$ thus $-p - 1 \leq x \leq 1 - p$. Now we are concerned with stability at the origin, therefore we would like x to be in an interval containing the origin. For this to happen, p has to be suitably chosen.

Choose p such that $-1 \leq p \leq 1$, that is $|p| \leq 1$. Therefore the region, in which we will be considering the Lyapunov function, is bounded on the right by the line $x = 1 - p$ and on the left by $x = -1 - p$.

We get instability if $(x+p)^2 > 1$

i.e. $\quad x + p > 1 , \quad x > 1 - p$

or $\quad x + p < -1 \ , \quad x < -1 - p \ .$

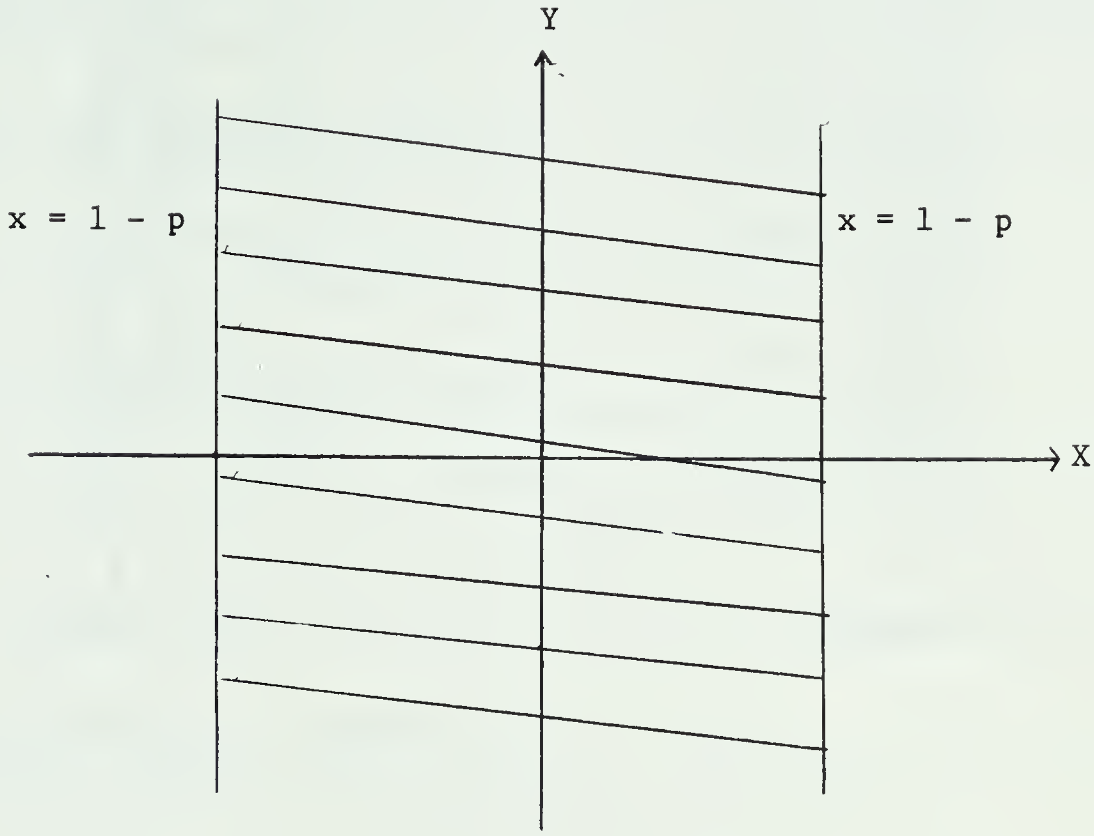


Figure 3.1

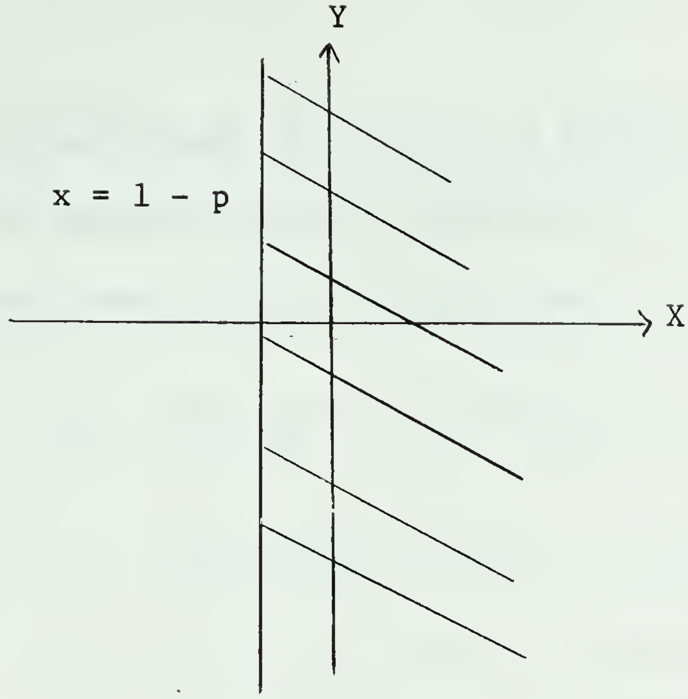
The shaded area is the region where the Lyapunov function is considered.

When $\mu > 0$, $V'(x,y) \leq 0$ if $(x+p)^2 \geq 1$, that is

(a) $x \geq 1 - p$ or

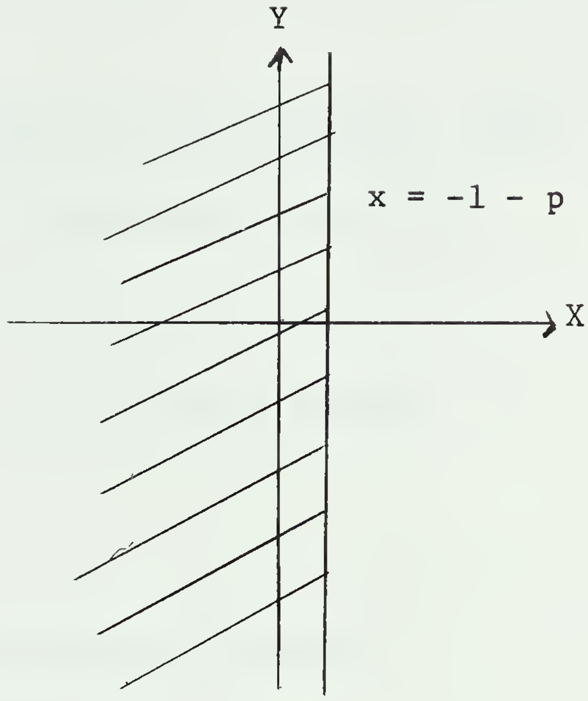
(b) $x \leq -1 - p$.

Now we are concerned with stability at the origin, so in case (a) we choose p such that $p \geq 1$ and in case (b) $p \leq 1$. We thus get as show on the following page.



p > 1

Figure 3.2



p < 1

Figure 3.3

We have instability if $(x+p)^2 < 1$

i.e. $$-1 < x + p < 1$$

$$-1 - p < x < 1 - p \quad .$$

7. The case $p(t)$ a non constant function of t .

Equation (3.3) can now be written as

$$x' = y$$

$$y' = -\mu((x+p(t))^2-1)y - (\mu p'(t)x+2\mu p(t)p'(t)+1)x \quad . \quad (3.10)$$

Consider the function,

$$V(x,y,t) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{\mu}{3}p'(t)x^3 + \mu p(t)p'(t)x^2 \quad . \quad (3.11)$$

We shall show that this is a Lyapunov function for the system (3.10) by imposing certain conditions on $p(t)$. We consider here only the case $\mu > 0$. We claim that

(a) $V(x,y,t)$ is defined for all $t \geq 0$ in some region.

(b) $V(0,0,t) = 0$ for $t \geq 0$.

(c) $V(x,y,t) \geq W(x,y)$ (a positive definite function).

Condition (b) is obviously true so we have now to find the region and also $W(x,y)$.

Now $V(x,y,t) = \frac{1}{2} y^2 + x^2(\frac{1}{2} + \frac{\mu}{3} p'(t)x + \mu p(t)p'(t))$.

Let us define $U(t,x,\mu)$ such that

$$U(t,x,\mu) \equiv \frac{1}{2} + \frac{\mu}{3} p'(t)x + \mu p(t)p'(t) .$$

If we choose, $p(t) \geq 0$, $0 \leq p'(t) < M$, and restrict x to be in the region $-a^2 \leq x$, $a^2 > 0$, then

$$U(t,x,\mu) \geq \frac{1}{2} + \frac{\mu}{3} p'(t)x$$

$$\geq \frac{1}{2} - \frac{\mu}{3} Ma^2 .$$

(Note since we are concerned with stability at the origin we want x

to be in some interval containing the origin).

Furthermore if we choose $0 < \delta^2 < \frac{1}{2}$, we get that $\frac{1}{2} - \frac{\mu}{3} Ma^2 \geq \delta^2$ if $a^2 \leq \frac{3/2 - 3\delta^2}{\mu M}$.

With these conditions imposed,

$$W(x,y) = \frac{1}{2} y^2 + \delta^2 x^2 .$$

We finally have to show that $V(x,y,t) \leq 0$, then expression (3.11) will be a Lyapunov function and the origin will be stable.

$$\begin{aligned} V'(x,y,t) &= xx' + yy' + \frac{\mu}{3} p''(t)x^3 + \mu p'(t)x^2x' + \mu x^2(p(t)p''(t)+(p'(t))^2) \\ &\quad + 2\mu p(t)p'(t)xx' \\ &= xy - \mu x^2y^2 - 2\mu p(t)xy^2 - \mu p^2(t)y^2 + \mu y^2 - \mu p'(t)x^2y \\ &\quad - 2\mu p(t)p'(t)xy - xy + \frac{\mu}{3} p''(t)x^3 + \mu p'(t)x^2y \\ &\quad + \mu x^2(p(t)p''(t)+(p'(t))^2) + 2\mu p(t)p'(t)xy \\ &= \mu x^2(p(t)p''(t)+(p'(t))^2) - \mu(p^2(t)-1)y^2 \\ &\quad - \mu(x^2y^2+2p(t)xy^2-p''(t)\frac{x^3}{3}). \end{aligned} \tag{3.13}$$

We will consider $-\mu(x^2y^2+2p(t)xy^2-\frac{x^3}{3}p''(t))$ as higher order terms. Equation (3.13) is of the form

$$V'(x,y,t) = A(t,\mu)x^2 - B(t,\mu)y^2 - \mu(x^2y^2+2p(t)xy^2-\frac{x^3}{3}p''(t))$$

where $A(t,\mu) = \mu(p(t)p''(t)+(p'(t))^2)$, $B(t,\mu) = \mu(p^2(t)-1)$. Since $\mu > 0$, for stability, it is necessary to have $A(t,\mu) \leq 0$, $B(t,\mu) \geq 0$.

Thus $p(t)p''(t) + (p'(t))^2 \leq 0$

$$p''(t) \leq \frac{-(p'(t))^2}{p(t)} \quad .$$

(Note that we needed $p(t)$ to be twice continuously differentiable) i.e. $p(t) \in C^2$.

$B(t,\mu) \geq 0$ implies $p^2(t) \geq 1$ therefore $p(t) \leq -1$ or $p(t) \geq 1$. Since we are considering $p(t)$ to be positive, we take only the latter condition and replace the original condition on $p(t)$, $0 \leq p(t)$, by $1 \leq p(t)$.

With these conditions imposed on $p(t)$, we see that expression (3.11) is a Lyapunov function for the system (3.10), for the higher order terms can be neglected as will be shown presently.

Since we are considering stability at the origin, we should take into account values of x and y in a small region around the origin. Our main purpose is to find out what effect, if any, do these higher order terms have on the nature of the stability. We claim that

they can be neglected under certain conditions.

The term $-\mu x^2y^2$ obviously causes no trouble for it is always negative for positive μ. We will look at $-2p(t)xy^2 + \frac{x^3}{3} p''(t)$. Consider $|x| < \varepsilon_1$, $|y| < \varepsilon_2$ $\varepsilon_1, \varepsilon_2 > 0$ and very small, let $\varepsilon = \max(\varepsilon_1, {}_2\varepsilon)$, we now have that

$$-2x(p(t)y^2 + \frac{x^2}{6} p''(t)) \leq -2\varepsilon_1(p(t)\varepsilon_2^2 + \frac{\varepsilon_1^2}{6} p''(t))$$

$$\leq k\varepsilon^3 \qquad (k \text{ a constant})$$

Now $k\varepsilon^3$ is small compared with $A(t,\mu)x^2 - B(t,\mu)y^2$ which is of the form $k_1\varepsilon^2$ provided ε is sufficiently small.

In the light of the previous discussion, we can now state the following theorem.

<u>Theorem 3.14</u>. Given (i) $p(t) \in C^2$, $1 \leq p(t)$

(ii) $0 \leq p'(t) < M$

(iii) $0 < \delta^2 < \frac{1}{2}$, $\mu > 0$

(iv) $x \geq -a^2$, $a^2 \leq \frac{3/2 - 3\delta^2}{\mu M}$.

The function $V(x,y,t)$ defined by expression (3.11) is a suitable Lyapunov function for the system (3.10).

A typical $p(t)$ satisfying the conditions of the theorem

is $p(t) = 3 - e^{-t}$.

We should note that we can find many functions $p(t)$ which satisfy the conditions of Theorem 3.14. There also exists various Lyapunov functions for the system (3.10) and these are obtained by imposing the necessary conditions on $p(t)$.

Although the case $\mu < 0$ will not be dealt with here, it is possible to find a Lyapunov function for this case and formulate a theorem similar to Theorem 3.10.

Also, the system (3.10) , really points out the fact that the "new" differential equation formed by the transformation is more complicated than the original. This can be seen by comparing system (3.10) and system (3.8).

It should be noted too, that in finding the Lyapunov function for this nonautonomous case, the positive definite function $W(x,y)$ played a role, whether indirectly, in determining the region of stability.

Finally, it should be realised that there may be values of μ , positive or negative, in this non-autonomous case and a function V which could present an unstable situation at the origin. It is possible that we could have

stability only in a region bounded away from the origin.

We will now discuss the simple pendulum equation.

CHAPTER IV

THE TRANSFORMED PENDULUM EQUATION

1. A simple pendulum consists of a mass m supported by a weightless rod of length ℓ pivoted at the top. The position of the mass is given by the angle x that the rod makes with the vertical. The equation of motion is given by

$$x'' + \alpha \sin x = 0 \tag{4.1}$$

were $\alpha = \frac{g}{\ell}$ and g is the acceleration due to gravity.

In this chapter, we will consider equation (4.1) and two others then make a transformation and study the resulting equations. The second equation is

$$x'' + cx' + \alpha \sin x = 0 \ . \tag{4.2}$$

This is the damped simple pendulum equation and the last equation is the forced damped simple pendulum equation

$$x'' + cx' + \alpha \sin x = \varepsilon f(t) \ . \tag{4.3}$$

In these three equations $\alpha > 0$, $c > 0$ and in equation (4.3),

$f(t+T) = f(t)$.

Before going into the discussion of these three equations, we will discuss nonlinear conservative systems. This material can be found in Minorsky [8], also in Butenin [3].

2. Fundamental properties of non-linear conservative systems.

The simpliest differential equation for such a system is

$$x'' + f(x) = 0 \tag{4.4}$$

which is equivalent to

$$\begin{aligned} x' &= y \\ y' &= -f(x) \end{aligned} \tag{4.5}$$

and the differential equation of the integral curves is

$$\frac{dy}{dx} = -\frac{f(x)}{y}, \tag{4.6}$$

which shows that the integral curves have a horizontal tangent at points x_i which are roots of $f(x) = 0$, provided $y \neq 0$ at these points. As to singular points, they require simultaneously $y = 0$, $f(x) = 0$. The energy integral is

$$\frac{1}{2} y^2 + \Pi(x) = h \tag{4.7}$$

where $\Pi(x) = -\int_0^x f(x)dx$. One can consider $\frac{1}{2} y^2 = \frac{1}{2} x'^2$ as the kinetic energy and $\Pi(x)$ as the potential energy, h the total energy expresses the fact that the system is conservative. For a given h, equation (4.7) represents a trajectory in the phase plane which exists as long as $h - \Pi(x) > 0$. A position of equilibrium corresponds to the case $y = 0$, $\Pi'(x) = 0$. The latter condition implies the differential equation has an extremum at the equilibrium point. Now equation (4.7) is equivalent to $\frac{1}{2} y^2 = h - \Pi(x)$. Let us examine the auxillary xz plane on which we shall construct $z = \Pi(x)$, (see Figure 4.1) and draw the line $z = h$. Read motion takes place only for those x located to the left of the point of intersection of $z = \Pi(x)$ and the line $z = h$. Now $y/\sqrt{2} = \pm \sqrt{h - \Pi(x)}$, it follows on the phase plane $(x, y_1 = y/\sqrt{2})$ the integral curves will have the form shown in Figure 4.2,

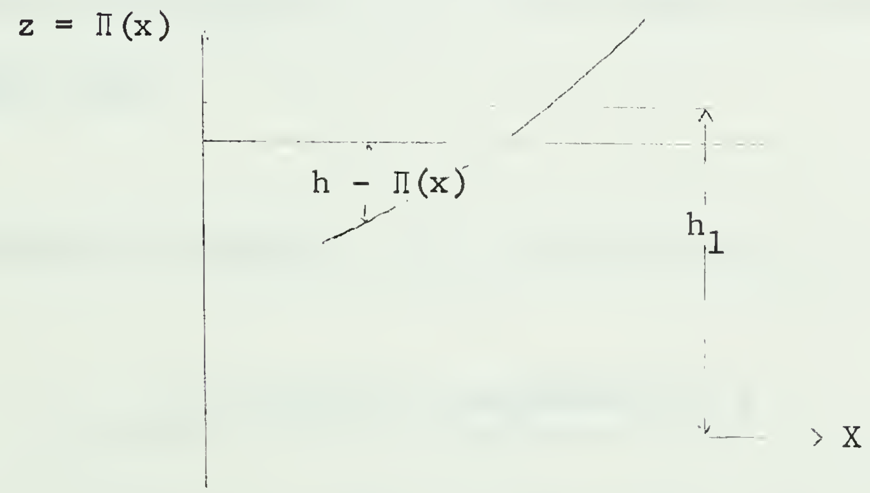


Figure 4.1.

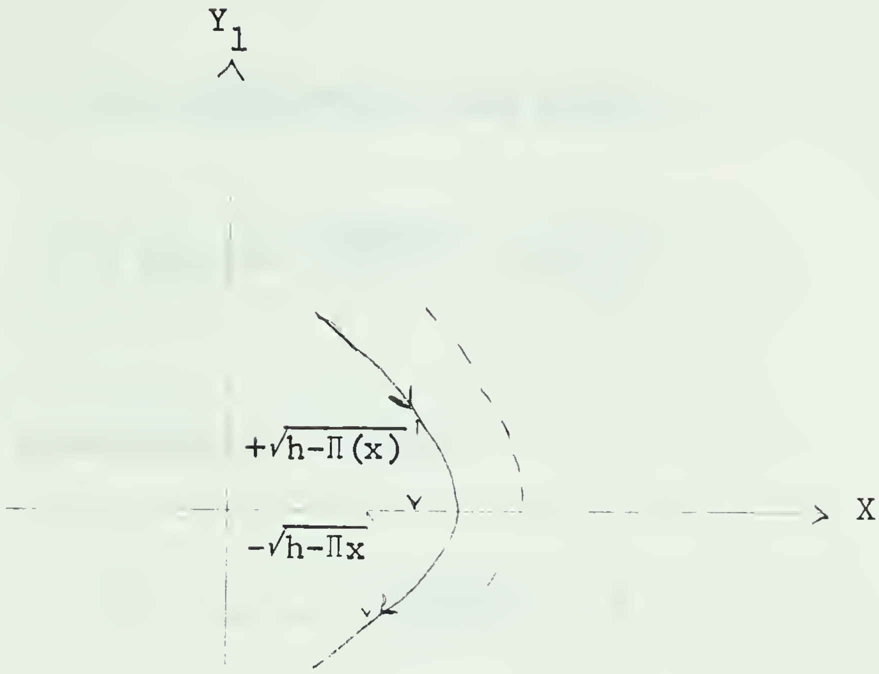


Figure 4.2

The Duffing equation discussed in Chapter II is an example of a conservative system.

As discussed in Minorsky [8], the stability of equilibrium is associated with the extremum values of $\Pi(x)$, the potential energy. When $\Pi(x)$ is a minimum, the curves in the neighbourhood of the origin are ellipse so the origin is a centre and when $\Pi(x)$ is a maximum, the curves in the neighbourhood of the origin are hyperbolas, the origin is a saddle point.

3. Returning to equation (4.1) and letting $x = y + p(t)$ we get

$$y'' + p''(t) + \alpha \sin(y+p(t)) = 0$$

$$y'' - \alpha \sin p(t) + \alpha \sin(y+p(t)) = 0$$

$$y'' + \alpha(\sin(y+p(t))-\sin p(t)) = 0$$

$$y'' + \alpha(2 \cos (\frac{y+2p(t)}{2}) \quad \sin \frac{y}{2}) = 0 \quad .$$

Without loss of generality, we consider

$$x'' + 2\alpha \cos (\frac{x+2p(t)}{2}) \sin \frac{x}{2} = 0 . \qquad (4.8)$$

In this case $\dfrac{dy}{dx} = \dfrac{-2\alpha \cos(\frac{x+2p(t)}{2}) \sin \frac{x}{2}}{y}$

We have a conservative system here. When $p(t) \equiv 0$, we get

$$\frac{dy}{dx} = - \frac{\alpha \sin x}{y}$$

$$ydy = - \alpha \sin xdx$$

$$\frac{y^2}{2} = h - \alpha \int \sin x \, dx$$

$$\Pi(x) = \alpha \int \sin x \, dx = - \alpha \cos x \ .$$

Therefore $y^2/2 = h + \alpha \cos x$, $\Pi(x)$ has isolated minima for $x = 0, \pm 2\pi, \pm 4\pi, \ldots$. The minima correspond to stable equilibrium states in the phase plane. They are centres. When $x = \pm\pi, \pm 3\pi, \ldots$, we have isolated maxima (saddle points).

For $h < -\alpha$, real motion of the pendulum does not exist.

For $-\alpha < h < \alpha$, the trajectories are closed and surround singular points which are centres. Closed trajectories correspond to periodic solutions of the pendulum.

For $h = \alpha$, the trajectories are separatrices which intersect at saddle points. For $h > \alpha$, the trajectories are divergent. But the x-coordinate is periodic, the motion of the representative point along the trajectories correspond to a clockwise or counter-clockwise motion of the pendulum about the suspension point. When $p(t) \equiv n\pi$, for n even, $\cos(\frac{x}{2} + p(t)) = \cos\frac{x}{2}$ and the singularities are the same as before. For n odd $\cos(\frac{x}{2} + p(t)) = -\cos\frac{x}{2}$. In this case we have maxima at $0, 2\pi, 4\pi, \ldots$, and hence saddle point where we had centre and vice versa.

4. <u>The case when $p(t) \equiv p$, a constant.</u>

Equation (4.8) can be written as

$$x'' + 2\alpha(\cos\frac{x}{2}\cos p - \sin\frac{x}{2}\sin p)\sin\frac{x}{2} = 0$$

$$x'' + \alpha\cos p \sin x - 2\alpha\sin^2\frac{x}{2}\sin p = 0$$

$$x'' + \alpha\cos p\sin x - \alpha(1-\cos x)\sin p = 0$$

$$x'' + \alpha(\cos p\sin x + \sin p\cos x) - \alpha\sin p = 0.$$

Or $\quad x' = y$

$$y' = -\alpha(\cos p \sin x + \sin p \cos x - \sin p)$$

So $\quad \dfrac{dy}{dx} = \dfrac{-\alpha(\cos p \sin x + \sin p \cos x - \sin p)}{y}$

i.e. $\quad ydy = (-\alpha \cos p \sin x - \alpha \sin p \cos x + \alpha \sin p)dx$

$$\frac{y^2}{2} = \alpha \cos p \cos x - \alpha \sin p \sin x + \alpha x \sin p \ .$$

In this case

$$\Pi(x) = -\int -\alpha \cos p \sin x - \alpha \sin p \cos x + \alpha \sin p \ dx$$

$$= -\alpha \cos p \cos x + \alpha \sin p \sin x - \alpha x \sin p$$

$$= -\alpha \cos(p+x) - \alpha x \sin p \ .$$

A position of equilibrium corresponds to the case when $\Pi'(x) = 0$, that is when

$$\sin (x+p) - \sin p = 0,$$

and so

$$p + x = p + 2\pi n$$

$$x = 2n\pi \ .$$

Or when

$$p + x = \pi - (p-2\pi n)$$

$$x = (2n+1)\pi - 2p$$

i.e.

$$x = n\pi - 2p \qquad\qquad (n \ \text{odd}).$$

Now $\Pi''(x) = \alpha\cos(x+p)$ and when $\Pi''(x) > 0$, we have a minimum (a centre); for $\Pi''(x) < 0$, we have a maximum (a saddle point).

If

$$x = 2\pi n, \quad \Pi''(x) = \alpha\cos(p+x) = \alpha\cos p \ .$$

If

$$\begin{aligned} x = n\pi - 2p, \quad \Pi''(x) = \alpha\cos(p+x) &= \alpha\cos(p+n\pi-2p) \\ &= \alpha\cos(n\pi-p) \\ &= (-1)^n\alpha\cos p \\ &= -\alpha\cos p \ . \end{aligned}$$

So if $\cos p > 0$, we have centres at $x = 2\pi n$ for all n. At this

time we have saddle points at $n\pi - 2p$ for n odd. The converse happens when $\cos p < 0$.

5. The case $p(t)$ a non constant function of t.

Recall

$$x'' + 2\alpha(\cos \frac{(x+2p(t))}{2} \sin \frac{x}{2}) = 0. \tag{4.8}$$

Or

$$x' = y$$

$$y' = -2\alpha(\cos \frac{(x+2p(t))}{2} \sin \frac{x}{2}). \tag{4.9}$$

We consider the function,

$$V(x,y,t) = \frac{1}{2} y^2 + \alpha \frac{x^2}{2} \cos p(t), \tag{4.10}$$

This will be shown to be a Lyapunov function for the system (4.9) with suitable restrictions on $p(t)$.

Firstly,

(i) $V(x,y,t)$ is defined for $t \geq 0$.

(ii) $V(0,0,t) = 0$ for $t \geq 0$.

That these two conditions hold is obvious. We further claim that $V(x,y,t)$ dominates $W(x,y) = \frac{1}{2} y^2 + \alpha k \frac{x^2}{2}$ where k will be specified

later.

$$V'(x,y,t) = yy' + \alpha \cos p(t)xx' - \alpha \frac{x^2}{2} p'(t) \sin p(t)$$

$$= -2\alpha y \cos \frac{1}{2} (x+2p(t))\sin \frac{x}{2} + \alpha \cos p(t)xy - \alpha \frac{x^2}{2} p'(t)\sin p(t)$$

$$= -\alpha y[\sin(x+p(t))-\sin p(t)] + \alpha xy \cos p(t) - \alpha \frac{x^2}{2} p'(t)\sin p(t)$$

$$= -\alpha y[\sin x \cos p(t)+\cos x \sin p(t)-\sin p(t)] + \alpha xy \cos p(t)$$

$$- \alpha \frac{x^2}{2} p'(t) \sin p(t)$$

$$= -\alpha y \cos p(t)[\sin x - x] - \alpha y \sin p(t)[\cos x-1]$$

$$- \alpha \frac{x^2}{2} p'(t) \sin p(t)$$

$$= -\alpha y \cos p(t)(- \frac{x^3}{3!} + o(x^5)) - \alpha y \sin p(t)[- \frac{x^2}{2} + o(x^4)]$$

$$- \alpha \frac{x^2}{2} p'(t) \sin p(t)$$

$$= \frac{\alpha}{6} x^3 y \cos p(t) - \alpha x^2 \sin p(t)(\frac{p'(t)}{2} - \frac{y}{2}) - \alpha y(\cos p(t)o(x^5)$$

$$+ \sin p(t)o(x^4))$$

$$= - x^2(\frac{-\alpha xy \cos p(t)}{6} + \frac{\alpha p'(t) \sin p(t)}{2}) + \alpha \frac{x^2}{2} y \sin p(t)$$

$$- \alpha y(\cos p(t)o(x^5)+\sin p(t)o(x^4)) .$$

This is of the form $-x^2(-B(t)xy+A(t))$ + higher order terms.
Here $A(t) = \frac{\alpha p'(t) \sin p(t)}{2}$, $B(t) = \frac{\alpha \cos p(t)}{6}$ and the higher order terms are

$$\alpha x^2 y \sin p(t) - \alpha y(\cos p(t) o(x^5) + \sin p(t) o(x^4)) .$$

For stability, it is necessary that

$$A(t) - B(t)xy > 0$$

i.e.

$$xy < \frac{A(t)}{B(t)} .$$

We also impose the condition that

$$A(t)B(t) > 0$$

for if not we have $xy < n$, n negative and since we are considering stability at the origin, we would not want this to occur.

Thus $\frac{\alpha p'(t) \sin p(t)}{2} \cdot \frac{\alpha \cos p(t)}{6} > 0$. We choose $p(t)$ such that

$$0 < \delta \leq \sin p(t) \leq 1 - \delta' \qquad 0 < \delta < 1,\ 0 < \delta' < 1$$

Then $0 < \arc\sin \delta \leq p(t) \leq \arc\sin (1 - \delta')$

$$0 < \cos \arc\sin (1 - \delta') \leq \cos p(t) \leq \cos \arc\sin \delta < 1 .$$

(Note we can now see that the k we need in our $W(x,y)$ is $k = \cos \arc\sin (1 - \delta')$. Thus $V(x,y,t) \geq \frac{1}{2} y^2 + \frac{\alpha}{2} \cos \arc\sin (1 - \delta')x^2$).

We also choose $p'(t)$ such that

$$q > p'(t) > 0$$

(q will be chosen suitably small) and

$$\lim_{t \to \infty} p'(t) = 0 .$$

Then

$$xy < \frac{+\dfrac{\alpha p'(t) \sin p(t)}{2}}{\dfrac{\alpha}{6} \cos p(t)}$$

$$= 3p'(t) \tan p(t).$$

Now $$\frac{\delta}{\cos \arc\sin \delta} \leq \tan p(t) \leq \frac{1 - \delta'}{\cos \arc\sin(1 - \delta')} .$$

So $$0 < 3p'(t)\tan p(t) \leq \frac{3q(1 - \delta')}{\sqrt{2\delta' - \delta'^2}} .$$

Therefore $xy < \dfrac{3q(1-\delta')}{\sqrt{2\delta' - \delta'^2}}$. If we choose $q = \sqrt{2\delta' - \delta'^2}$ we get that

$$xy < 3(1-\delta') = \ell \qquad \ell > 0$$

With this restriction on xy we have stability at the origin.

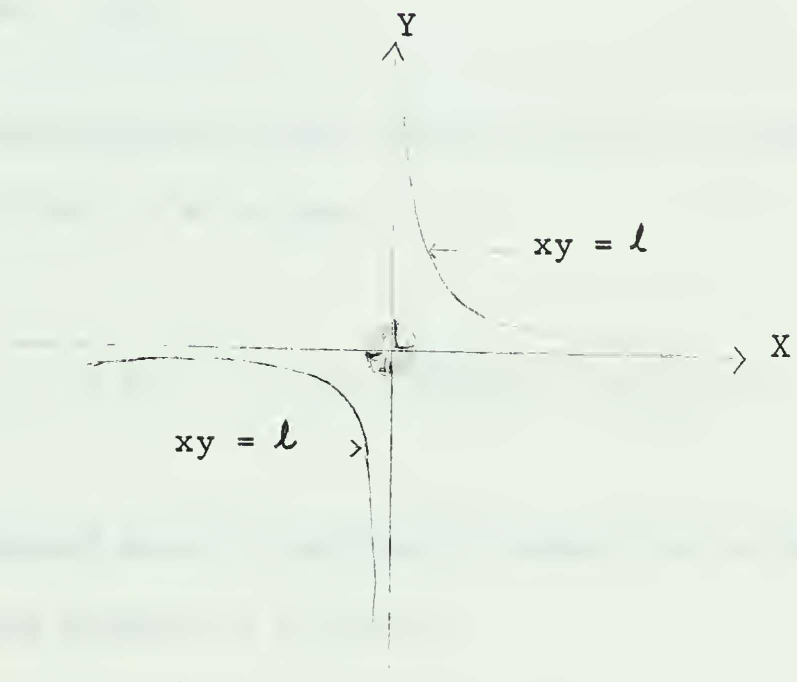


Figure 4.3.

(Shaded portion shows stable region.)

It remains to discuss the higher order terms. We will show that these are easily dispersed of when we are close to the origin. It is obvious that in the higher order terms, $-\alpha y(\cos p(t) o(x^5) + \sin p(t) o(x^4))$ can be neglected for $|x| < \varepsilon_1$, $|y| < \varepsilon_2$. The term $\alpha x^2 y \sin p(t)$ is of the form $C\varepsilon^3$ which is small when compared with $-x^2(A(t)-B(t)xy)$ which is of the form $C_1\varepsilon^2$ provided ε is sufficiently small, $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$.

In the light of the preceeding discussion, we state the following theorem.

<u>Theorem 4.11</u> Given (i) $p(t) \varepsilon C'$,

(ii) $0 < \delta \leq \sin p(t) \leq 1 - \delta' \quad 0 < \delta < 1, 0 < \delta' < 1$

(iii) $0 < p'(t) < q, \quad \lim_{t \to \infty} p'(t) = 0$.

The function $V(x,y,t)$ defined by (4.10) is a suitable Lyapunov function for the system (4.9).

6. Performing the usual transformation on equation (4.2) and changing y's for x's we get

$$x'' + cx' + 2 \left(\cos \left(\frac{x+2p(t)}{2}\right) \sin \frac{x}{2}\right) = 0 . \tag{4.12}$$

We are considering here a dissipative system for we have large deviations of the pendulum subject to friction.

<u>Cape $p(t) \equiv 0$</u>. Equation (4.12) can now be written as

$$x' = y$$

$$y' = -cy - \alpha \sin x . \tag{4.13}$$

In the phase plane $\frac{dy}{dx} = -c - \frac{\alpha \sin x}{y}$. The singularities are $x = k\pi$, $y = 0$ (k is any integer positive or negative) and correspond to stable or unstable equilibria according as k is even or odd. Taking the first term in the expansion of $\sin x$, the system (4.13) becomes

$$x' = y$$

$$y' = -\alpha x - cy \ .$$

Here $\Delta = c^2 - 4\alpha$, $p = -c$, $q = \alpha$. Referring to Table 1, Chapter I, we see that if $c^2 > 4\alpha$, we have a stable node at the origin. If $c^2 < 4\alpha$, we have a stable spiral.

The singularities which were centres in the case where there was no damping become spiral or nodal points. We saw in the earlier case that at $x = \pi$, the singularity was a saddle point. In the case with damping, we again get a saddle point for the singularity is not affected by the damping term cx' . The singularities at $x = k\pi$ are stable if k is even, unstable if k is odd. The tangents to the paths are vertical on the x-axis and horizontal on the curve

$$y = -\frac{\alpha \sin x}{c} \ .$$

On the basis of these facts, one can construct a qualitative graph of the arrangement of the trajectories. From the figures below, it follows that the system cannot have periodic motion.

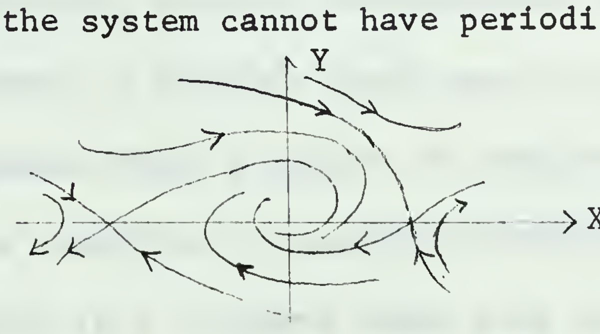


Figure 4.4 Stable spiral

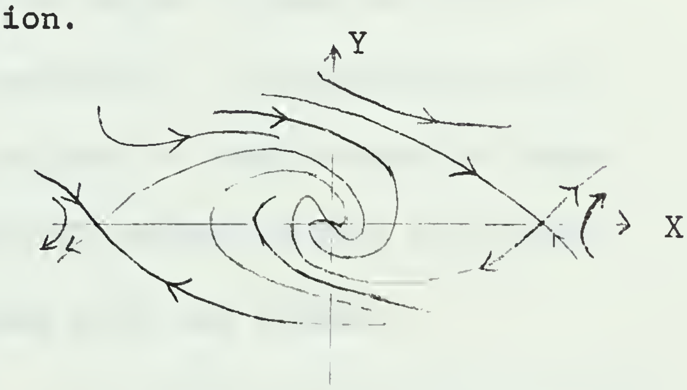


Figure 4.5 Stable node

It is interesting to note that part of the phase plane between $-\pi$ and π can be thought of as laid out on the surface of a cylinder.

We will briefly discuss the cylindrical phase space. This is fully dealt with in Andronow and Chaikin [1].

The description of the behaviour of a dynamical system by means of a phase space requires a one-one correspondence between the states of the system and the points of the space. Up to now, we have been investigating physical systems whose phase space can be represented by a plane. In general, however, this plane will not be satisfactory, as is seen by considering the system of the ordinary pendulum (the system with which we are dealing). Here the state of the system is determined by the angle of deviation from the equilibrium position and the velocity of the pendulum. If we restrict ourselves to motion which does not exceed one revolution, no difficulties will arise if the phase space is taken as a plane. But when the angle of deviation is increased by a multiple of 2π, we obtain a state identical with the original state. In the phase "plane", we would have an unlimited number of points corresponding to the same physical state of the system. Since the necessary one-one correspondence does not hold, a plane is not in general a suitable phase space for a pendulum. In particular, for systems whose position is completely defined by their deviation angle, the condition of one-one correspondence is satisfied when the phase space is a cylinder whose axis coincides with the y-axis.

In the case of a cylindrical phase space there may occur two

kinds of closed paths.

(a) Closed paths of first kind. These are the ordinary closed paths surrounding the equilibrium states without going around the cylinder and they are completely analogous to the closed paths in the phase plane.

(b) Closed paths of second kind. They go around the cylinder without surrounding the equilibrium states.

Obviously both kinds of closed paths correspond to periodic motion.

7. Case $p(t) \equiv p$ a constant.

The equation with which we are dealing now is

$$x' = y$$

$$y' = -cy - \alpha \sin(x+p) + \alpha \sin p .$$

The isocline (the locus of points where the paths have a given slope) $\frac{dy}{dx} = 0$ has the equation

$$y = \frac{-\alpha \sin(x+p) + \alpha \sin p}{c} \tag{4.14}$$

and represents a displaced sinusoidol. It crosses the x-axis only when

$\alpha \sin p < \alpha$, that is, $\sin p < 1$. Also $\frac{dy}{dx} > 0$ between the sinusoid and and the x axis and $\frac{dy}{dx} < 0$ elsewhere for

$$\frac{dy}{dx} > 0 \quad \text{if} \quad \frac{-cy - \alpha \sin(x+p) + \alpha \sin p}{y} > 0$$

$$y > 0 : y < \frac{\alpha \sin p - \alpha \sin(x+p)}{c}$$

$$y < 0 : y > \frac{\alpha \sin p - \alpha \sin(x+p)}{c} .$$

The singular points are at $\sin(x+p) = \sin p$, $y = 0$ and as we saw before are the points

$$B_k : x = 2n\pi \qquad y = 0 .$$

$$A_k : x = n\pi - 2p , \quad y = 0 .$$

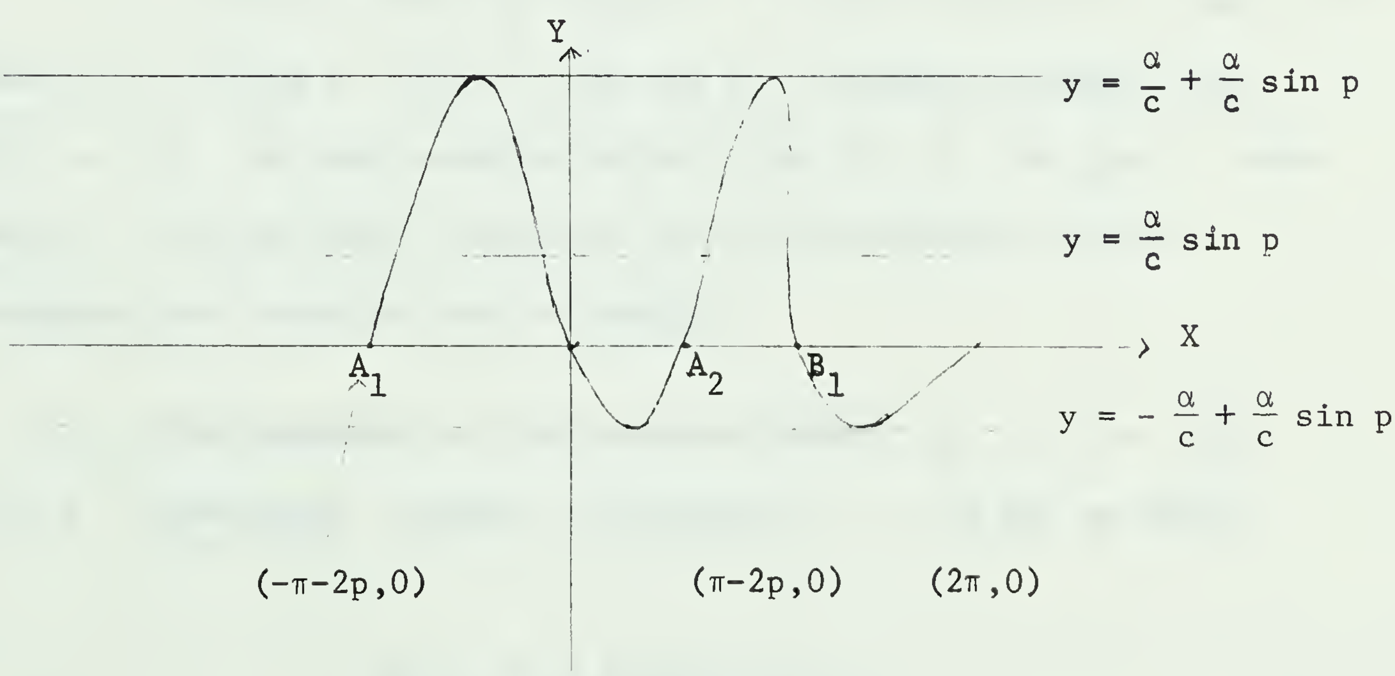


Figure 4.6. Displaced Sinusoid.

We consider first the points B_k. Let $x = 2n\pi + \eta$ ($\eta > 0$ and very small) and develop $\sin(x+p)$ in powers of η. If we keep only terms of first order in η, the behaviour of the system in the neighbourhood of B_k is given by

$$\frac{dy}{d\eta} = \frac{-cy - \alpha\eta \cos p + \ldots}{y} .$$

Thus the coefficient matrix is

$$\begin{pmatrix} 0 & 1 \\ -\alpha \cos p & -c \end{pmatrix} .$$

We now have that $\Delta = c^2 - 4\alpha \cos p$, $\tilde{p} = -c$, $q = \alpha \cos p$.

From the table in Chapter I, we have a node if $\alpha \cos p > 0$ and $c^2 - 4\alpha \cos p > 0$, $c^2 > 4\alpha \cos p$. The node is stable for $\tilde{p} = -c < 0$. We have a stable spiral point if $c^2 < 4\alpha \cos p$. Note when $c = 0$, we have a centre as was seen previously when we examined the equation with no damping.

We consider now the singular points $A_k : x = n\pi - 2p$, $y = 0$. Developing $\sin(x+p)$ in powers of η, we get as above

$$\frac{dy}{d\eta} = \frac{-cy + \alpha\eta \cos p + \ldots}{y} .$$

In this case we have a saddle point. Again when $c = 0$, we get a saddle

point as in the case with no damping.

Andronow and Chaikin [1] discusses in detail the existence of periodic solution for such a system.

8. <u>The case $p(t)$ a nonconstant function of t.</u>

$$x' = y$$

$$y' = -cy - 2\alpha(\cos(\frac{x+2p(t)}{2})\sin\frac{x}{2}).$$

We use the same Lyapunov function as given by equation (4.10) for in this case

$$V'(x,y,t) = -cy^2 - x^2(\frac{-\alpha xy\cos p(t)}{6} + \frac{\alpha p'(t)\sin p(t)}{2})$$

$$+ \alpha x^2 y \sin p(t) - \alpha y(\cos p(t) o(x^5) + \sin p(t) o(x^4)).$$

The only new term is $-cy^2$ and this does not affect the sign of V' for it is always negative, so we again have stability. Thus Theorem 4.11 goes through in this case.

9. <u>The case when $x = p(t,\varepsilon)$ is a solution</u>

The last case to be discussed is

$$x'' + cx' + \alpha\sin x = \varepsilon f(t). \qquad (4.3)$$

Here we let $x = p(t,\varepsilon)$ be a solution. We suppose $\varepsilon > 0$, small, and that $p(t,\varepsilon)$ admits the expansion,

$$p(t,\varepsilon) = p_o(t) + \varepsilon p_1(t) + \varepsilon^2 p_2(t) + \ldots = p_o(t) + \varepsilon p_1(t) + o(\varepsilon).$$

After transforming we find that equation (4.3) can be written as the system

$$x' = y$$

$$y' = - cy - 2\alpha \cos \frac{(x+2p(t,\varepsilon))}{2} \sin \frac{x}{2} . \qquad (4.15)$$

Now $y' = - cy - 2\alpha[\cos p(t,\varepsilon) - \frac{x}{2} \sin p(t,\varepsilon) + o(x)][\frac{x}{2} + o(x)]$ for

$$\cos(\frac{x}{2} + p(t,\varepsilon)) = \cos \frac{x}{2} \cos p(t,\varepsilon) - \sin \frac{x}{2} \sin p(t,\varepsilon)$$

$$= (1 - \frac{x^2}{2} + o(x^2))\cos p(t,\varepsilon) - (\frac{x}{2} - (\frac{x}{2})^3 + \ldots)\sin p(t,\varepsilon)$$

$$= \cos p(t,\varepsilon) - \frac{x}{2} \sin p(t,\varepsilon) + o(x).$$

We are considering here only the first terms in the expansions of $\sin x$ and $\cos x$

$$y' = -cy - 2\alpha[\cos(p_o(t)+\varepsilon p_1(t)+o(\varepsilon)) - \frac{x}{2} \sin(p_o(t)+\varepsilon p_1(t)+o(\varepsilon)+o(x)][\frac{x}{2} +o(x)]$$

$$= -cy - 2\alpha[\cos p_o(t)-\varepsilon p_1(t)\sin p_o(t)- \frac{x}{2} \sin p_o(t)+ \frac{x}{2}\, o(\varepsilon)$$

$$- \frac{x}{2} p_1(t) \cos p_o(t) + (x)][\frac{x}{2} + o(x)] .$$

The last line follows because, if ϵ is sufficiently small,

$$\sin(p_o(t)+\varepsilon p_1(t)+o(\varepsilon)) = \sin p_o(t)\cos(\varepsilon p_1(t)+o(\varepsilon)) + \cos p_o(t)\sin(\varepsilon p_1(t)+o(\varepsilon))$$

$$= \sin p_o(t)(1-o(\varepsilon)) + \varepsilon p_1(t)\cos p_o(t).$$

Similarly

$$\cos(p_o(t)+\varepsilon p_1(t)+o(\varepsilon)) = \cos p_o(t)\cos(\varepsilon p_1(t)+o(\varepsilon)) - \sin p_o(t)\sin(\varepsilon p_1(t)+o(\varepsilon))$$

$$= \cos p_o(t)(1-o(\varepsilon)) - \varepsilon p_1(t)\sin p_o(t)$$

$$y' = -cy - 2\alpha[\frac{x}{2} \cos p_o(t) - \varepsilon \frac{x}{2} p_1(t)\sin p_o(t) - \frac{x^2}{4} \sin p_o(t) - \varepsilon \frac{x^2}{4} p_1(t)\cos p_o(t)$$

$$+ \frac{x^2}{4} o(\varepsilon) + o(x^2)]$$

$$y' = -cy - \alpha x(\cos p_o(t) - \frac{x}{2} \sin p_o(t)) + \varepsilon\alpha x p_1(t)(\sin p_o(t) + \frac{x}{2} \cos p_o(t))$$

$$- \alpha \frac{x}{2} o(\varepsilon) - 2\alpha o(x^2).$$

The problem now is to find a suitable Lyapunov function for this system.

Consider

$$V(x,y,t) = \frac{1}{2} y^2 + \frac{\alpha}{2} x^2 \cos p_o(t) - \varepsilon\alpha p_1(t)\sin p_o(t) \frac{x^2}{2} . \qquad (4.16)$$

Choose $p_o(t)$ such that

$$0 < \delta \leq \sin p_o(t) \leq 1 - \delta' \qquad 0 < \delta < 1, \quad 0 < \delta' < 1$$

i.e.

$$0 < \arcsin \delta \leq p_o(t) \leq \arcsin(1-\delta')$$

$$0 < \cos \arcsin(1-\delta') \leq \cos p_o(t) \leq \arcsin \delta < 1$$

$$0 < \sqrt{1 - (1-\delta')^2} \leq \cos p_o(t) \leq \sqrt{1 - \delta^2} .$$

Let $\qquad 0 \leq p_1(t) < M, \quad p_1'(t) \geq 0, \quad p_o'(t) \geq 0 \quad .$

Then $V(x,y,t) \geq \frac{1}{2} y^2 + (\frac{\alpha}{2} \sqrt{2\delta' - \delta'^2} - \varepsilon\alpha M(1-\delta'))\frac{x^2}{2}$). We want $W(x,y)$ to be positive definite so it is necessary that

$$\alpha\sqrt{2\delta' - \delta'^2} - \varepsilon\alpha M(1-\delta') > 0$$

$$M < \frac{\sqrt{2\delta' - \delta'^2}}{\varepsilon(1-\delta')} \quad .$$

It is now clear that

(i) $V(x,y,t)$ is defined for all $t \geq 0$

(ii) $V(0,0,t) = 0$ for $t \geq 0$

(iii) $V(x,y,t) \geq W(x,y) = \frac{1}{2} y^2 + \alpha(\sqrt{2\delta' - \delta'^2} - \varepsilon M(1-\delta'))\frac{x^2}{2}$.

It remain to show that $V'(x,y,t) \leq 0$.

$$V'(x,y,t) = yy' + \alpha xx' \cos p_o(t) - \frac{\alpha}{2} x^2 p_o'(t)\sin p_o(t)$$

$$- \varepsilon\alpha \frac{x^2}{2} \{p_1(t)p_o'(t)\cos p_o(t)+p_1'(t)\sin p_o(t)\}$$

$$- \varepsilon\alpha p_1(t)\sin p_o(t)xx'$$

$$= -cy^2 - \alpha xy \cos p_o(t) + \alpha \frac{x^2}{2} y \sin p_o(t) + \varepsilon\alpha xy\, p_1(t) \sin p_o(t)$$

$$+ \varepsilon\alpha\frac{x^2}{2} yp_1(t)\cos p_o(t) - \alpha \frac{x}{2} y\, o(\varepsilon) - 2\alpha y o(x^2)$$

$$+ \alpha xy \cos p_o(t) - \frac{\alpha}{2} x^2 p_o'(t)\sin p_o(t) - \varepsilon\alpha xyp_1(t)\sin p_o(t)$$

$$- \varepsilon\alpha \frac{x^2}{2} \{p_1(t)p_o'(t)\cos p_o(t)+p_1'(t)\sin p_o(t)\}$$

$$= -cy^2 - \frac{\alpha}{2} x^2 p_o'(t)\sin p_o(t) - \varepsilon\alpha \frac{x^2}{2} \{p_1(t)p_o'(t)\cos p_o(t)+p_1'(t)\sin p_o(t)\}$$

$$+ \alpha \frac{x^2}{2} y \sin p_o(t) + \varepsilon\alpha\frac{x^2}{2} yp_1(t)\cos p_o(t) - \alpha \frac{x}{2} y o(\varepsilon)$$

$$- 2\alpha y o(x^2) .$$

We will consider the last four terms as higher order terms. The first four terms in the expression for V' are negative so we do not have to worry about them.

Since our only concern is stability in a small region around the origin say $|x| < \varepsilon_1$, $|y| < \varepsilon_2$, the last four terms are of the form $c_1\varepsilon^3$ which can be neglected when compared with terms of the form $c_2\varepsilon^2$ (the first four terms) provided ε is sufficiently small. We thus have stability at the origin.

Theorem 4.17 $V(x,y,t)$ given by (4.16) is a Lyapunov function for the system (4.15) provided

(i) $0 < \delta \leq \sin p_o(t) \leq 1 - \delta'$, $0 < \delta < 1$, $0 < \delta' < 1$.

(ii) $p_o'(t) \geq 0$, $0 \leq p_1(t) < M$, $p'(t) \geq 0$.

(iii) $\varepsilon > 0$ and sufficiently small.

BIBLIOGRAPHY

1. Andronow, A.A. and Chaikin, E.E. "Theory of Oscillations", Princeton University Press, Princeton, New Jersey, 1949.

2. Brand, L. "Differential and Difference Equations", John Wiley and Sons.

3. Butenin, N.V. "Elements of the Theory of Non-Linear Oscillations", Blaisdell Publishing Company.

4. Cesari, L. "Asymtotic Behaviour and Stability Problems in Ordinary Differential Equations", Springer Verlag, 1963.

5. Coddington, E. and Levison, N. "Theory of Ordinary Differential Equations", McGraw Hill Book Company, New York, 1955.

6. Hurewicz, W. "Lectures on Ordinary Differential Equations", M.I.T. Press, Cambridge, Mass., 1958.

7. La Salle, J. and Lefschetz, S. "Stability by Lyapunov's Direct Method with Applications", Academic Press, 1961.

8. Minorsky, N. "Non-linear Oscillations", D.Van Nostrand Company Inc., Princeton, New Jersey.

9. Ritzen, P. and Rose, N. "Differential Equations with Applications", McGraw Hill Book Company.

10. Sanchez, D.A. "Ordinary Differential Equations--Stability Theory; An Introduction", W.H. Freeman and Company, San Francisco.

11. Stoker, J.J. "Non-linear Vibrations in Mechanical and Electrical Systems", Interscience Publishers Inc., New York, 1950.

APPENDIX

PERTURBATIONS ANS STABILITY

If we consider two systems $x' = f(x)$ ... (1) and $x' = f(x,\varepsilon)$ and if we have some information about the stability of the first system, it is of interest to investigate the manner in which the stability of the second system depends on ε .

When we made the transformation on $x' = y' + p'(t)$ and considered the case $p(t) = p$, a constant, we were looking at the system $x' = F(x,p)$ ... (2) , where p could be considered as a perturbation parameter.

Intuitively, we would expect that for sufficiently small p , the stability behaviour of the original system should not be changed.

This was in fact borne out when we considered the Duffing equation. Certain small value of p had the effect of shifting the paths but not changing the singularities.

We should note now that $F(x,0) = f(x)$ and $F(0,p) = 0$. If (1) has a solution $\phi(x)$ and (2) has a solution $\Phi(x,p)$ then $\Phi(x,0) = \phi(x)$ since F is C^1 in x and p .

Let us now examine $x' = f(x)$ with the equilibrium state at the origin. The perturbed system is $x' = F(x,p)$. Let Q be a closed and bounded set containing the origin and Q_o a subset of Q . Let Φ be a solution of (2) and P be the set of perturbations

such that $|p| < \delta$ (if we were considering the non-autonomous case we would want $|p(x,t)| < \delta$ for all $t \geq 0$ and all x). Suppose solutions which start in Q_o remain in Q .

The concept of stability is relative to the sets Q , Q_o and the number δ . Suppose we consider Q_o as the original ball in which we have stability, we see that δ could determine how close the stability region of the perturbed system would be to Q_o (i.e. whether it would be a superset of Q_o or be a subset of it).

This question leads to uniform boundedness of solutions relative to Q_o and the class of perturbations P . The bound should not only exist but be sufficiently small. Thus for practical results a knowledge of the size of Q_o and δ should be known.

What this all means is that for very small δ , we would find, the stability region is not appreciably changed. This has been noticed in the cases dealt with in this thesis.